Osnovne formule iz Matematike II

Dio tablica integrala.

1. $\int u^a \mathrm{d}u = \dfrac{u^{a+1}}{a+1} + C,\ a \neq -1.$
2. $\int u^{-1} \mathrm{d}u = \int \dfrac{\mathrm{d}u}{u} = \int \dfrac{u'}{u} \mathrm{d}x = \ln|u| + C.$
3. $\int a^u \mathrm{d}u = \dfrac{a^u}{\ln|a|} + C; \int e^u \mathrm{d}u = e^u + C.$
4. $\int \sin \mathrm{d}u = -\cos u + C.$
5. $\int \cos \mathrm{d}u = \sin u + C.$
6. $\int \dfrac{1}{\cos^2 u} \mathrm{d}u = \operatorname{tg} u + C.$
7. $\int \dfrac{1}{\sin^2 u} \mathrm{d}u = -\operatorname{ctg} u + C.$
8. $\int \dfrac{\mathrm{d}u}{u^2 + a^2} = \dfrac{1}{a} \operatorname{arc\,tg} \dfrac{u}{a} + C.$
9. $\int \dfrac{\mathrm{d}u}{u^2 - a^2} = \dfrac{1}{2a} \ln \left| \dfrac{u-a}{u+a} \right| + C.$
10. $\int \dfrac{\mathrm{d}u}{\sqrt{a^2 - u^2}} = \arcsin \dfrac{u}{a} + C.$
11. $\int \dfrac{\mathrm{d}u}{\sqrt{u^2 + a}} = \ln|u + \sqrt{u^2 + a}| + C.$

Newton-Leibnizova formula.

$$\int_a^b f(u)\mathrm{d}u = \int f(u)\mathrm{d}u \Big|_a^b = F(u)|_a^b = F(b) - F(a), \text{ gdje je } F'(u) = f(u).$$

Osobine određenih integrala.

1. $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x.$
2. $\int_a^a f(x)\mathrm{d}x = 0.$
3. $\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$
4. $\int_a^b [f_1(x) + f_2(x) - f_3(x)]\mathrm{d}x = \int_a^b f_1(x)\mathrm{d}x + \int_a^b f_2(x)\mathrm{d}x - \int_a^b f_3(x)\mathrm{d}x.$
5. $\int_a^b c\, f(x)\mathrm{d}x = c \int_a^b f(x)\mathrm{d}x.$

Smjena promjenjivih u određenom integralu.

$$\int_a^b f(x)\mathrm{d}x = \left| \begin{array}{ll} x = \varphi(t) & x = a \Rightarrow a = \varphi(\alpha) \Rightarrow t = \alpha \\ dx = \varphi'(t)\mathrm{d}t & x = b \Rightarrow b = \varphi(\beta) \Rightarrow t = \beta \end{array} \right| = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\mathrm{d}t = \int_\alpha^\beta h(t)\mathrm{d}t$$

Nepravi integrali. $\displaystyle\int_a^{+\infty} f(x)\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\mathrm{d}x, \int_{-\infty}^{b} f(x)\mathrm{d}x = \lim_{a\to-\infty} \int_a^b f(x)\mathrm{d}x$,...

Računanje površine ravne figure. U zavisnosti od izgleda slike: $P = \int_a^b f(x)\mathrm{d}x$, $P = \int_c^d g(y)\mathrm{d}y$, $P = -\int_a^b f(x)\mathrm{d}x$, $P = \int_a^b [\eta(x) - \mu(x)]\mathrm{d}x$, $P = \int_c^d [g(y) - h(y)]\mathrm{d}y$, ...

Zapremina rotacionog tijela. Ako, kriva data u parametarskom obliku $C : \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \leq t \leq t_2 \end{cases}$ rotira oko x-ose, zapremine se računa po formuli

$$V_x = \pi \int_{t_1}^{t_2} [\mu(t)]^2 |\eta'(t)| \mathrm{d}t.$$

Ista kriva ako rotira oko y-ose, $V_y = \pi \int_{t_1}^{t_2} [\eta(t)]^2 |\mu'(t)| \mathrm{d}t$. Iz ove dvije formule, za funkcije $y = f(x)$ i $x = g(y)$, slijedi $V_x = \pi \int_a^b [f(x)]^2 \mathrm{d}x$ i $V_y = \pi \int_c^d [g(y)]^2 \mathrm{d}y$.

Dužina luka krive. $C : \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \leq t \leq t_2 \end{cases}$, $\ell = \int\limits_{t_1}^{t_2} \sqrt{[\eta'(t)]^2 + [\mu'(t)]^2} \mathrm{d}t$;

$C : \begin{cases} y = f(x) \\ a \leq x \leq b \end{cases}$, $\ell = \int\limits_a^b \sqrt{1 + [y'(x)]^2} \mathrm{d}x$; $C : \begin{cases} x = g(y) \\ c \leq y \leq d \end{cases}$, $\ell = \int\limits_c^d \sqrt{1 + [g'(y)]^2} \mathrm{d}y$;

Komplanacija obrtne površi. Površina omotača tijela dobijenog rotacijom krive $C : \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \leq t \leq t_2 \end{cases}$, oko x-ose, se računa po formuli: $P = 2\pi \int\limits_{t_1}^{t_2} |\mu(t)| \sqrt{[\eta'(t)]^2 + [\mu'(t)]^2} \mathrm{d}t$;

$C : \begin{cases} y = f(x) \\ a \leq x \leq b \end{cases}$, $P = 2\pi \int\limits_a^b |f(x)| \sqrt{1 + [f'(x)]^2} \mathrm{d}x$; ...

Funkcije dvije nazavisno promjenjive. ...

Parcijalni izvodi f-ja više pomjenjivih. $z = f(x, y)$, $z'_x = \dfrac{\partial z}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ ...

Diferenciranje funkcija više promjenjivih. $u = f(x, y, z)$, $du = \dfrac{\partial u}{\partial x} \mathrm{d}x + \dfrac{\partial u}{\partial y} \mathrm{d}y + \dfrac{\partial u}{\partial z} \mathrm{d}z$ ...

Diferenciranje složenih funkcija. ...

Parcijalni izvodi višeg reda složenih funkcija. ...

Ekstremne vrijednosti f-ja dvije promjenjive. ...

Uslovni ekstremi f-ja dvije promjenjive. ...

Jednačina tangentne ravni i jednačina normale na površ. Ako je S u obliku $F(x, y, z) = 0$

$$\alpha : \; F'_x(x_0, y_0, z_0)(x - x_0) + F'_y(x_0, y_0, z_0)(y - y_0) + F'_z(x_0, y_0, z_0)(z - z_0) = 0$$

$$n : \; \frac{x - x_0}{F'_x(x_0, y_0, z_0)} = \frac{y - y_0}{F'_y(x_0, y_0, z_0)} = \frac{z - z_0}{F'_z(x_0, y_0, z_0)}$$

Dvojni integrali.

$$\iint\limits_D f(x, y) = \int\limits_a^b \mathrm{d}x \int\limits_{g(x)}^{h(x)} f(x, y) \mathrm{d}y = \int\limits_a^b \left[\int\limits_{g(x)}^{h(x)} f(x, y) \mathrm{d}y \right] \mathrm{d}x,$$

$$\iint\limits_D f(x, y) = \int\limits_c^d \mathrm{d}y \int\limits_{\eta(y)}^{\mu(y)} f(x, y) \mathrm{d}x = \int\limits_c^d \left[\int\limits_{\eta(y)}^{\mu(y)} f(x, y) \mathrm{d}x \right] \mathrm{d}y \ldots$$

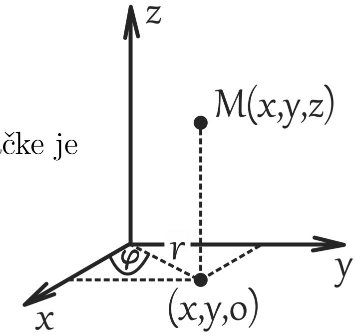
Smjena promjenjivih u dvojnim integralima. Za prelazak sa pravougaonih na polarne koordinate koristimo smjene $\begin{cases} x = r\cos(\varphi) \\ y = r\sin(\varphi) \\ \mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\varphi \end{cases}$, poopštene plarne koordinate su oblika $\begin{cases} x = a\,r\cos(\varphi), \; (a > 0) \\ y = b\,r\sin(\varphi), \; (b > 0) \\ \mathrm{d}x\mathrm{d}y = a\,b\,r\mathrm{d}r\mathrm{d}\varphi \end{cases}$, a za proizvoljne smjene $\begin{cases} x = \eta(u, v) \\ y = \mu(u, v) \\ \mathrm{d}x\mathrm{d}y = |J|\mathrm{d}u\mathrm{d}v \end{cases}$, gdje je J Jakobijan,

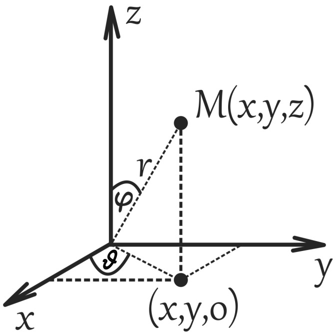
$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Trojni integrali. ...

Računanje trojnih integrala uvođenjem cilindričnih i sfernih koordinata.

Na cilindrične koordinate prelazimo pomoću $\left\{\begin{array}{c} x = r\cos(\varphi) \\ y = r\sin(\varphi) \\ z = z \\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = r\mathrm{d}r\mathrm{d}\varphi\mathrm{d}z \end{array}\right.$, opis tačke je





Za prelazak sa pravougaonih na sferne koordinate koristimo sljedeće smjene $\left\{\begin{array}{c} x = r\sin(\varphi)\cos(\theta) \\ y = r\sin(\varphi)\sin(\theta) \\ z = r\cos(\varphi) \\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = r^2\sin(\varphi)\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta \end{array}\right.$, (opis tačke je prikazan na slici lijevo).

Primjena dvostrukih integrala.

$$(a)\ P = \iint\limits_D \mathrm{d}x\mathrm{d}y. \qquad (b)\ V = \iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y.$$

Primjena trostrukih integrala. $(a)\ V = \iiint\limits_\Omega \mathrm{d}x\mathrm{d}y\mathrm{d}z.$

$(b)\ T(x_T, y_T, z_T),\ \ x_T = \dfrac{1}{V}\iiint\limits_\Omega x\mathrm{d}x\mathrm{d}y\mathrm{d}z, y_T = \dfrac{1}{V}\iiint\limits_\Omega y\mathrm{d}x\mathrm{d}y\mathrm{d}z, z_T = \dfrac{1}{V}\iiint\limits_\Omega z\mathrm{d}x\mathrm{d}y\mathrm{d}z.$

Krivoliniski integral prve vrste (po luku).

$$C: \left\{\begin{array}{c} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{array}\right., \quad \int\limits_C f(x,y)\mathrm{d}s = \int\limits_{t_1}^{t_2} f(\eta(t),\mu(t))\sqrt{(\eta'(t))^2 + (\mu'(t))^2}\mathrm{d}t.$$

$$C: \left\{\begin{array}{c} y = f(x) \\ a \le x \le b \end{array}\right., \quad \int\limits_C z(x,y)\mathrm{d}s = \int\limits_a^b z(x,f(x))\sqrt{1 + (f'(x))^2}\mathrm{d}x.$$

Primjena krivoliniskog integrala prve vrste - Računanje površine cilindrične površi.

$$C: \left\{\begin{array}{c} F(x,y) = 0 \\ z = 0 \end{array}\right., \quad P = \int\limits_C z(x,y)\mathrm{d}s.$$

Krivoliniski integral druge vrste (po koordinatama).

$$C: \left\{\begin{array}{c} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{array}\right., \quad \int\limits_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \int\limits_{t_1}^{t_2} [P(\eta(t),\mu(t))\eta'(t) + Q(\eta(t),\mu(t))\mu'(t)]\mathrm{d}t.$$

$$C: \left\{\begin{array}{c} y = f(x) \\ a \le x \le b \end{array}\right., \quad \int\limits_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \int\limits_a^b [P(x,f(x)) + Q(x,f(x))f'(x)]\mathrm{d}x.$$

Krivoliniski integral druge vrste *ovisi o smjeru puta integracije*.

Formula Greena.

$$\int\limits_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \iint\limits_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y.$$

Primjena krivoliniskog integrala druge vrste - Računanje površine ravne figure.

$$P = \frac{1}{2}\int\limits_C x\mathrm{d}y - y\mathrm{d}x.$$

Nezavisnost krivoliniskog integrala od vrste konture. Određivanje primitivnih funkcija.

..., $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, ..., $\mathrm{d}u(x,y) = \dfrac{\partial u}{\partial x}\mathrm{d}x + \dfrac{\partial u}{\partial y}\mathrm{d}y$, ...

Površinski integral prve vrste.

D projekcija od S: $z=\eta(x,y)$ na $x0y$ - $\iint\limits_S f(x,y,z)\mathrm{d}S = \iint\limits_D f(x,y,\eta(x,y))\sqrt{1+\left(\frac{\partial \eta}{\partial x}\right)^2+\left(\frac{\partial \eta}{\partial y}\right)^2}\mathrm{d}x\mathrm{d}y.$

E projekcija od S: $y=\mu(x,z)$ na $x0z$ - $\iint\limits_S f(x,y,z)\mathrm{d}S = \iint\limits_E f(x,\mu(x,z),z)\sqrt{1+\left(\frac{\partial \mu}{\partial x}\right)^2+\left(\frac{\partial \mu}{\partial z}\right)^2}\mathrm{d}x\mathrm{d}z.$

F projekcija od S: $x=\gamma(y,z)$ na $y0z$ - $\iint\limits_S f(x,y,z)\mathrm{d}S = \iint\limits_F f(\gamma(y,z),y,z)\sqrt{1+\left(\frac{\partial \gamma}{\partial y}\right)^2+\left(\frac{\partial \gamma}{\partial z}\right)^2}\mathrm{d}y\mathrm{d}z.$

Površinski integral druge vrste.

Ako je integral oblika $\iint\limits_S P(x,y,z)\mathrm{d}y\mathrm{d}z+Q(x,y,z)\mathrm{d}x\mathrm{d}z+R(x,y,z)\mathrm{d}x\mathrm{d}y$ obično ga podjelimo na tri dijela $\iint\limits_S P(x,y,z)\mathrm{d}y\mathrm{d}z$, $\iint\limits_S Q(x,y,z)\mathrm{d}x\mathrm{d}z$, $\iint\limits_S R(x,y,z)\mathrm{d}x\mathrm{d}y$. Neka je $\vec{n}=(\cos\alpha,\cos\beta,\cos\gamma)$ vektor normale na površinu S, gdje su α, β i γ uglovi koje vektor normale zaklapa sa x, y i z osom. Tada

$$I_1=\iint\limits_S P(x,y,z)\mathrm{d}y\mathrm{d}z = \left|\begin{array}{c} S: x=\eta(y,z), \\ \text{neka je } D \text{ projekcija od } S \text{ na } y0z \text{ ravan,} \\ \text{neka je } \alpha \text{ ugao koji vektor} \\ \text{normale na } S \text{ zaklapa sa } x\text{-osom,} \end{array}\right| = \pm\iint\limits_D P(\eta(y,z),y,z)\mathrm{d}y\mathrm{d}z \text{ gdje}$$

vrijednost za $\pm$ zavisi od $\cos(\alpha)$ ($\cos(\alpha)>0$ stavljamo +, za $\cos(\alpha)<0$ stavljamo -, a za $\cos(\alpha)=0$ imamo $I_1=0$). Slično za I_2 i I_3

$$I_2=\iint\limits_S Q(x,y,z)\mathrm{d}x\mathrm{d}z = \left|\begin{array}{c} S: y=\mu(x,z), \\ \text{neka je } E \text{ projekcija od } S \text{ na } x0z \text{ ravan,} \\ \text{neka je } \beta \text{ ugao koji vektor} \\ \text{normale na } S \text{ zaklapa sa } y\text{-osom,} \end{array}\right| = \pm\iint\limits_E Q(x,\mu(x,z),z)\mathrm{d}x\mathrm{d}z.$$

$$I_3=\iint\limits_S R(x,y,z)\mathrm{d}x\mathrm{d}y = \left|\begin{array}{c} S: z=\delta(x,y), \\ \text{neka je } F \text{ projekcija od } S \text{ na } x0y \text{ ravan,} \\ \text{neka je } \gamma \text{ ugao koji vektor} \\ \text{normale na } S \text{ zaklapa sa } x\text{-osom,} \end{array}\right| = \pm\iint\limits_F R(x,y,\delta(x,y))\mathrm{d}x\mathrm{d}y.$$

Primjena površinskog integrala prve vrste - Izračunavanje površine dijela glatke površi.

$P=\iint\limits_S \mathrm{d}S=\iint\limits_D \sqrt{1+\left(\frac{\partial \eta}{\partial x}\right)^2+\left(\frac{\partial \eta}{\partial y}\right)^2}\mathrm{d}x\mathrm{d}y$, gdje je D projekcija od S: $z=\eta(x,y)$ na $x0y$ ravan.

Stoksova formula. ...

Formula Gauss-Ostrogradski.

$$\iint\limits_S P(x,y,z)\mathrm{d}y\mathrm{d}z+Q(x,y,z)\mathrm{d}x\mathrm{d}z+R(x,y,z)\mathrm{d}x\mathrm{d}y=\iiint\limits_\Omega \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Integrali ovisni o parametru.

$I(\alpha)=\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\mathrm{d}x \quad \Longrightarrow \quad I'(\alpha)=\int_{a(\alpha)}^{b(\alpha)} f'_\alpha(x,\alpha)\mathrm{d}x+b'(\alpha)f(b(\alpha),\alpha)-a'(\alpha)f(a(\alpha),\alpha).$

Ako granice a i b ne zavise od α tada $I'(\alpha)=\int\limits_a^b f'_\alpha(x,\alpha)\mathrm{d}x$.

Vektorska teorija polja. ...

Cirkulacija i fluks vektorskog polja.

$$C=\int\limits_c \vec{v}\,\mathrm{d}\vec{r}=\int\limits_c v_x\mathrm{d}x+v_y\mathrm{d}y+v_z\mathrm{d}z.$$

$$\Phi=\iint\limits_S \vec{v}\vec{n}\,\mathrm{d}S=\iint\limits_S v_x\mathrm{d}y\mathrm{d}z+v_y\mathrm{d}x\mathrm{d}z+v_z\mathrm{d}x\mathrm{d}z.$$